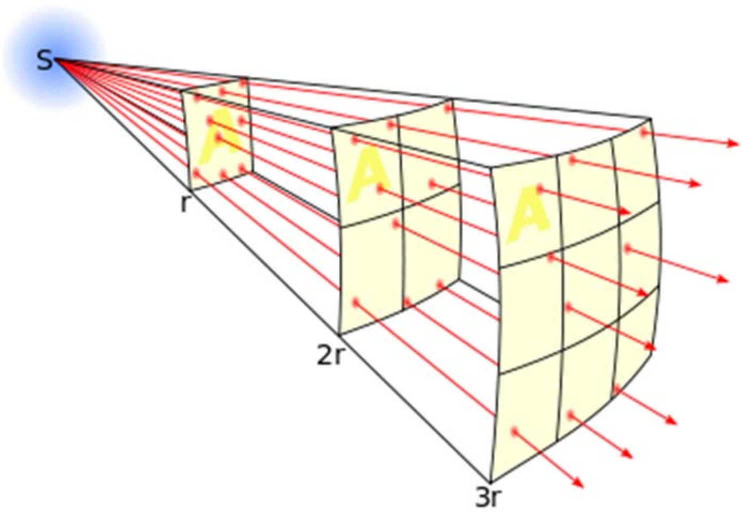
Seminary 5
Gravitation

UNIVERSAL ATTRACTION LAW

Inverse-square law. Which is the physical origin of the inverse square law (e.g. gravitational, electrostatic force)?

Answer: *The inverse-square law in Physics* $\propto\left(\frac{1}{r^2}\right)$, *is any physical law stating that a specified physical quantity or intensity is inversely proportional to the square of the distance from the source of that physical quantity. The fundamental cause for this can be understood as geometric dilution corresponding to point-source radiation into three-dimensional space.*



The inverse-square law generally applies when some force, energy, or other conserved quantity is evenly radiated outward from a point source in three-dimensional space. Since the surface area of a sphere (which is $4\pi r^2$ *is proportional to the square of the radius, as the emitted radiation gets farther from the source, it is spread out over an area that is increasing in proportion to the square of the distance from the source. Hence, the intensity of radiation passing through any unit area (directly facing the point source) is inversely proportional to the square of the distance from the point source. Gauss' law is similarly applicable, and can be used with any physical quantity that acts in accord to the inverse-square relationship.*

For a radiating source ***intensity ~ 1/distance***^{2} **.**

Gravitation is the attraction of two objects with mass. Newton's law states: **The gravitational attraction force between two-point masses is directly proportional to the product of their**

masses and inversely proportional to the square of their separation distance. The force is always attractive and acts along the line joining them. *If the distribution of matter in each body is spherically symmetric, then the objects can be treated as point masses (good approximation if the separation between the massive bodies is much larger compared to their size*).

Electrostatics *The force of attraction or repulsion between two electrically charged particles, in addition to being directly proportional to the product of the electric charges, is inversely proportional to the square of the distance between them; this is known as Coulomb's law.*

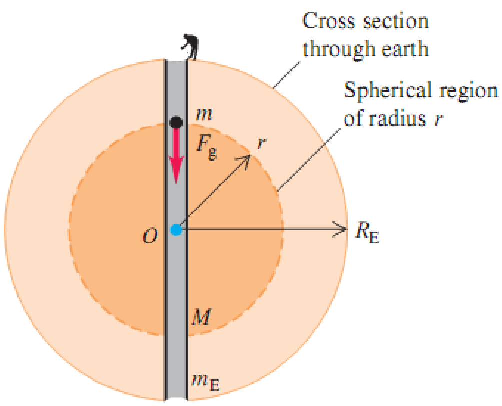
Waves, Light and other electromagnetic radiation

The intensity of light or other linear waves radiating from a point source (energy per unit of area perpendicular to the source) is inversely proportional to the square of the distance from the source; so an object (of the same size) twice as far away, receives only one-quarter the energy (in the same time period).

Gravity train

A gravity train would be a hypothetical means of transportation intended to go between two points on the surface of the Earth, following a straight tunnel that goes directly from one point to the other through the center of the Earth. Demonstrate that this train (mass m) could be left to accelerate using just the force of gravity, since, during the first half of the trip (from the point of departure until the middle), the downward pull towards the center of gravity would pull it towards the destination. During the second half of the trip, the acceleration would be in the opposite direction relative to the trajectory, but (ignoring the effects of friction) the speed acquired before would be enough to cancel this deceleration exactly (so that the train would reach its destination with speed equal to zero).

Answer:

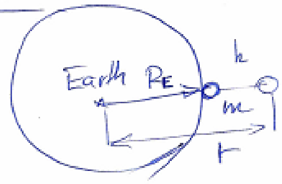


In a first step we calculate the variation of g with the distance r with respect to the surface of the Earth of radius R_E, in two situations:

(a) $r < R_E$ (object of mass m outside of the Earth).

(b) $r > R_E$ (object of mass m inside of the Earth).

① outside



$$w = F_g = \frac{G M_E m}{r^2} =$$

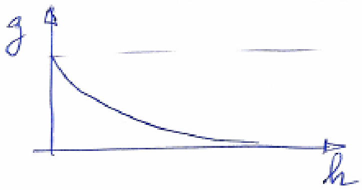
$$= G \frac{M_E m}{(R_P + h)^2}$$

$R_E = 6380\,km$

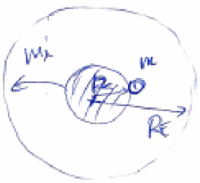
$$h = 0 \Rightarrow F_g = \underbrace{G \frac{M_E}{R_P^2}}_{g_0} m = m g_0$$

$g_0 = 9{,}80\,m/s^2$

$$\Rightarrow mg = m \underbrace{\frac{G M_E}{R_P^2}}_{g_0} \frac{R_P^2}{(R_P + h)^2} \Rightarrow \boxed{g = g_0 \left(\frac{R_E}{R_E + h}\right)^2}$$



② inside $r < R_E$



$$\rho = \frac{M_E}{V} = \frac{M_E}{\frac{4\pi}{3} R_E^3}$$

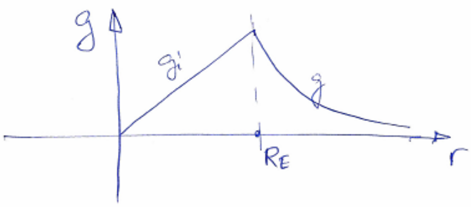
$$m_i = \rho V = \frac{M_E}{\cancel{\frac{4\pi}{3}} R_E^3} \cdot \cancel{\frac{4\pi}{3}} r^3$$

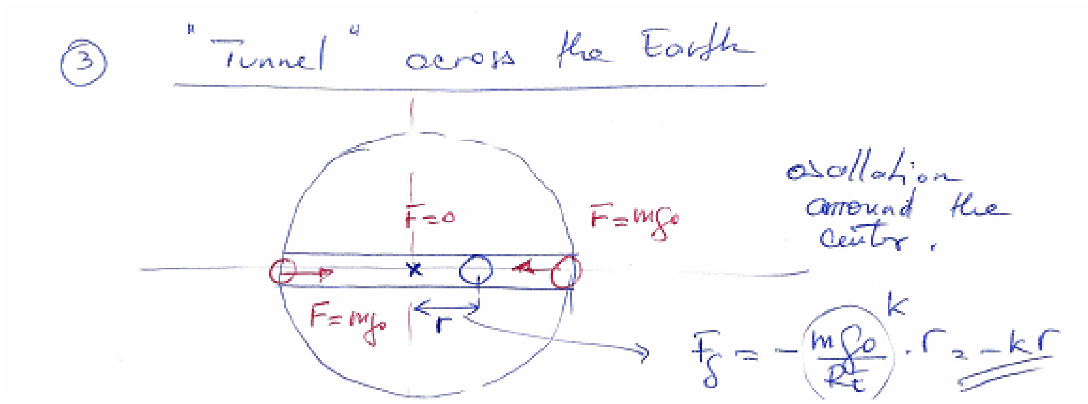
$$m_i = M_E \cdot \frac{r^3}{R_E^3}$$

$$F_g = G \cdot \frac{m_i m}{r^2} = \frac{G M_E}{\cancel{r^2}} \cdot \frac{r^{\cancel{3}}}{R_E^3} = \overbrace{\frac{G M_E}{R_E^2}}^{g_0} \cdot \frac{r}{R_E}$$

$$\Rightarrow \boxed{g_i = g_0 \frac{r}{R_E}}$$

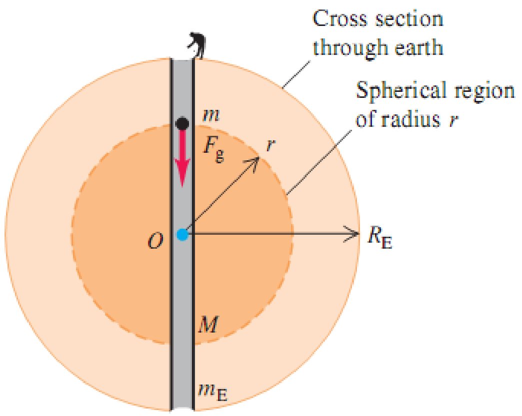
$r = 0 \Rightarrow g_i = 0$





Restoring force => oscillation

13.24 A hole through the center of the earth (assumed to be uniform). When an object is a distance *r* from the center, only the mass inside a sphere of radius *r* exerts a net gravitational force on it.

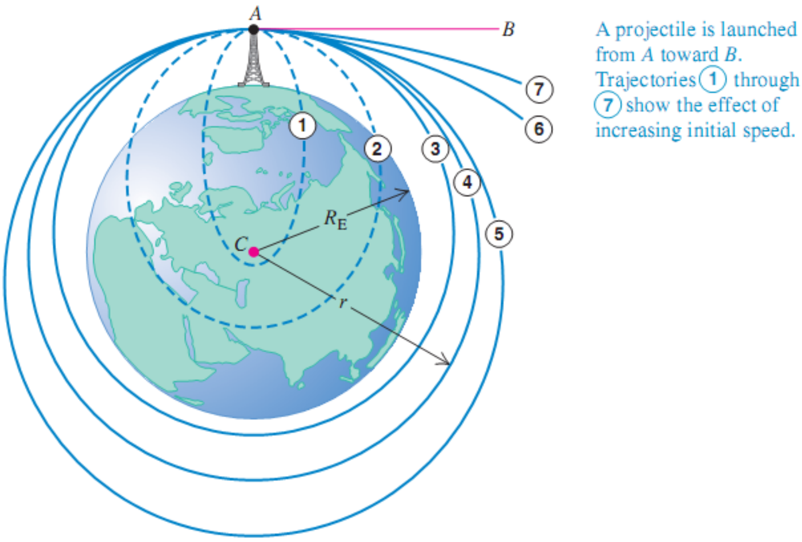


Motion of satellites

The satellites have a constant circular motion describing circular orbital movement around the Earth. The radius of the circular satellite orbit r is measured from the center of the Earth. Given the mass of the earth M_E and the universal attraction constant *G*.

a) Illustrate how, by increasing the initial speed an object launched from the surface of the Earth, one can change its trajectory from parabola to close circular orbit then open orbit.
b) Using Newton's universal law of gravitation and the definition of the centripetal force calculate the velocity of the satellite on a closed circular path of radius r and show that this does not depend on the satellite mass.
c) A 1000 kg satellite describes a circular orbit at 300km above the surface of the Earth (R_E=6380km). Calculate its speed, period and radial acceleration.

Answer:



Trajectories of a projectile launched from a great height (ignoring air resistance). Orbits 1 and 2 would be completed as shown if the earth were a point mass at *C*. (This illustration is based on one in Isaac Newton's *Principia.*)

a) *We launch a projectile from point A in the direction AB, tangent to the earth's surface. Trajectories 1 through 7 show the effect of increasing the initial speed. In trajectories 3 through 5 the projectile misses the earth and becomes a satellite. If there is no retarding force, the projectile's speed when it returns to point A is the same as its initial speed and it repeats its motion indefinitely.*

Trajectories 1 through 5 close on themselves and are called closed orbits. All closed orbits are ellipses or segments of ellipses; trajectory 4 is a circle, a special case of an ellipse. Trajectories 6 and 7 are open orbits. For these paths the projectile never returns to its starting point but travels ever farther away from the earth.

b) *satellite speed in a circular closed path motion*

Circular orbits => satellites => constant circular motion around the Earth.

Let the radius of the sattelite orbit R – measured from the center of the Earth.

=> acceleration of the satellite:

$a_{rad} = \frac{v^2}{R}$ allways directed towards the center of the Earth

By the law of gravitation (Newton's law), the net force (gravitational force) acting on the satellite of mass m has the magnitude:

$$F_g = G\frac{M_E m}{r^2}$$ and has the same direction as the acceleration

Newton 2nd law tells us $\Sigma \vec{F} = m\vec{a}$

$$\Rightarrow \quad \frac{G M_E m}{r^2} = m\frac{v^2}{r} \quad \Rightarrow$$

$$\boxed{v = \sqrt{\frac{G M_E}{r}}}$$ (circular orbit)

$\Rightarrow$ we can't choose the orbit radius r and the speed v independently; for a given value of r the v for a circular orbit is determined.

$\Rightarrow$ the motion of the satellite does not depend on its mass.

c)

(R) The radius of the satellite orbit is

$r = 6380\ km + 300\ km = 6680\ km = 6{,}68 \cdot 10^6\ m$

from $v = \sqrt{\frac{G M_E}{r}} = \sqrt{\frac{6{,}67 \cdot 10^{-11} \frac{N m^2}{kg^2} \cdot 5{,}97 \cdot 10^{24} kg}{6{,}68 \cdot 10^6\ m}}$

$v = 7720\ m/s$

$$T = \frac{2\pi r}{v} = \frac{2\pi\,(6{,}68\cdot 10^{6}\ m)}{7720\ m/s} = 5440\ s = 90{,}6\ min$$

The radial acceleration is:

$$a_{rad} = \frac{v^2}{r} = \frac{(7720\ m/s)^2}{6{,}68\cdot 10^{6}\ m} = 8{,}92\ m/s^2.$$

GRAVITATIONAL POTENTIAL ENERGY

An object of mass m is situated at a distance r with respect to the center of the Earth of mass M_E. a) a) Using the universal attraction law and the relation between force and potential energy calculate the potential energy of the mass m in the gravitational field.

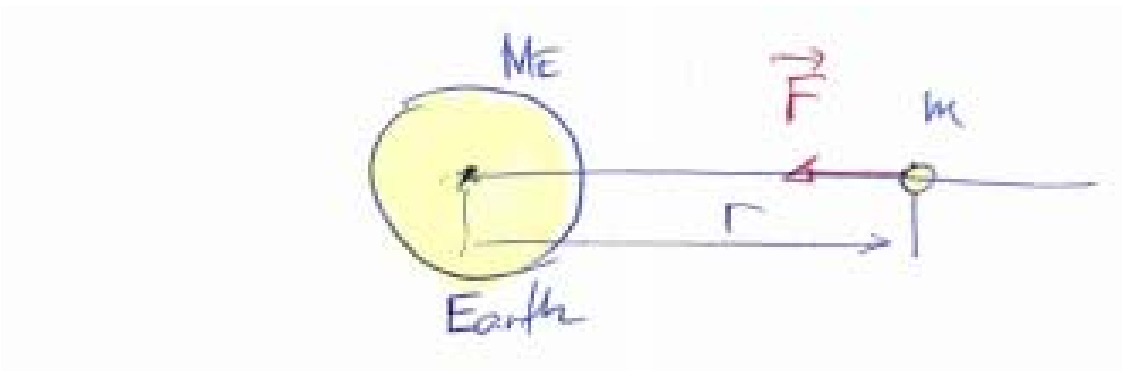
b) Show that when we are close to the surface of the Earth the above result reduces to the familiar U=mgy.

c) Using total energy conservation, calculate a minimum speed of a projectile launched vertically from the surface of the Earth to reach an altitude equal to the Earth radius R_E.

d) Based on same energy conservation, calculate the minimum projectile speed that allows to escape the Earth completely (and go to infinity). Discuss the results as a function of the mass of the planet.

Answer:

a) Gravitational potential energy



$$F(r) = -G\frac{M_E m}{r^2} \qquad F(r) = -\frac{dU}{dr} \Rightarrow$$

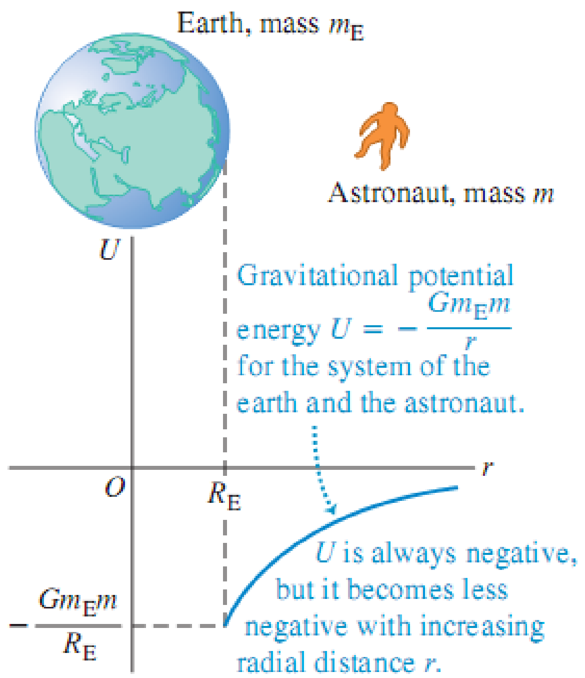
$$\int_r^{\infty} dU = -\int_r^{\infty} F(r)\, dr$$

$$\underline{U(\infty) = 0}$$

$$\Rightarrow \underbrace{U(\infty)}_{0} - U(r) = -\int_r^{\infty}\left(-\frac{GMm}{r^2}\right) dr$$

$$= -\frac{GMm}{r}\Big|_r^{\infty} = \frac{GMm}{r}$$

$$\Rightarrow \boxed{U(r) = -G\frac{M_E m}{r}}$$

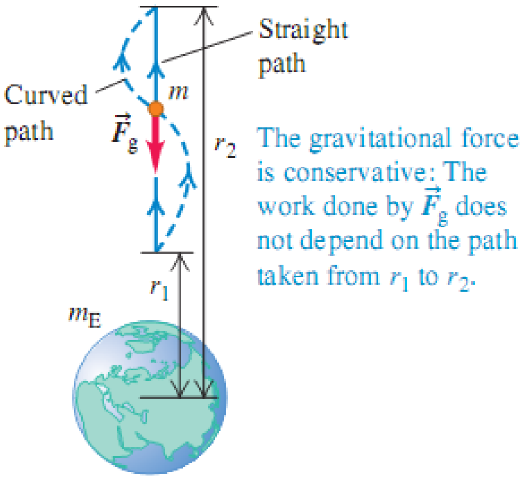


A graph of the gravitational potential energy U for the system of the earth (mass and an astronaut (mass m) versus the astronaut's distance r from the center of the earth.

b) *Close to the surface of the Earth*. From:

$$F_r = -\frac{Gm_E m}{r^2}$$

We consider a body of mass m outside the earth, and first compute the work done by the gravitational force when the body moves directly away from or toward the center of the earth from r_1 to r_2 as in Fig.



$$W_{grav} = \int_{r_1}^{r_2} F_r \, dr$$

=>

$$W_{grav} = -Gm_E m \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{Gm_E m}{r_2} - \frac{Gm_E m}{r_1}$$

<=>

$$\boxed{W_{grav} = Gm_E m \frac{r_1 - r_2}{r_1 r_2}}$$

If the body stays close to the earth, then in the denominator we may replace r1 and r2 and by the earth's radius R_E, so:

$$W_{grav} = Gm_E m \frac{r_1 - r_2}{R_E^2}$$

But:

$g = Gm_E / R_E^2$, so

$$W_{grav} = mg(r_1 - r_2)$$

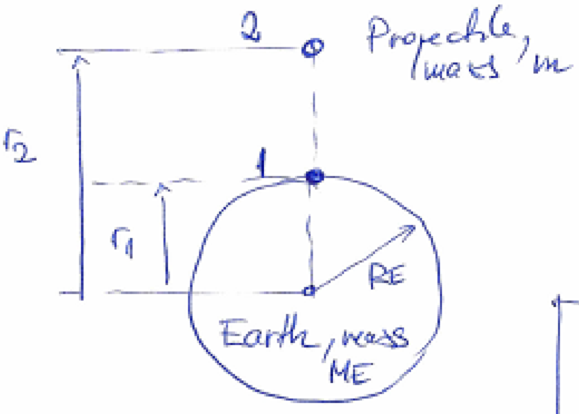
If we replace the r's by y's, we get the expression for the work done by a constant gravitational force. $W_{grav} = mg(y1 - y_2)$ see course.

⇨ **U = mgy** , that would be a particular case of

$$\boxed{U = -\frac{Gm_E m}{r} \quad \text{(gravitational potential energy)}}$$

for the case when we are very close to the surface of the Earth.

c) ***Projectile vertically launched to get altitude equal to R_E***:



The energy conservation equation:

$$\boxed{K_1 + U_1 = K_2 + U_2 \quad \text{with } U = -\frac{G M_E m}{r}}$$

The conservation law

a) $$\Rightarrow \frac{1}{2} m v_1^2 + \left(-\frac{G M_E m}{R_E}\right) = 0 + \left(-\frac{G M_E m}{2 R_E}\right)$$

gives the minimum projectile speed needed to shoot a shell straight up to a height above the Earth equal to the Earth radius R_E

$$\left(\begin{array}{l} r_2 = 2R_E \\ r_1 = R_E \end{array}\right)$$

$$\Rightarrow v_1 = \sqrt{\frac{G M_E}{R_E}} = \sqrt{\frac{6{,}67 \cdot 10^{-11} \frac{Nm^2}{kg^2} \cdot 5{,}97 \cdot 10^{24} kg}{6{,}38 \cdot 10^6 \ m}}$$

$$= 7900 \ m/s \approx 28400 \ km/h$$

d) Escape velocity

b) The escape speed is the minimum projectile speed that allows a shell to escape the Earth completely

$\Rightarrow \quad r_2 = \infty \;;\; U_2 = 0 \quad$ and $K_2 = 0$
$(v_2 = 0)$

$$\Rightarrow K_2 + U_2 = 0 = K_1 + U_1$$

$$\frac{1}{2} m v_1^2 + \left(-G\frac{M_E m}{R_E}\right) = 0 \Rightarrow$$

$$\boxed{v_1 = \sqrt{2\frac{GM_E}{R_E}}}$$

$$v_1 = \sqrt{\frac{2\,(6{,}67\cdot 10^{-11}\,Nm^2/kg^2)\cdot 5{,}97\cdot 10^{24}\,kg}{6{,}38\cdot 10^{6}\,m}} =$$

$$\boxed{v_1 = 1{,}12\cdot 10^4\,m/s} = 40200km/h$$

Discussion, Generalization:

$v_1 = \sqrt{\frac{2GM_E}{R_E}}$ does not depend on the mass of the shell or the direction of launch.

A modern spacecraft launched from Florida must attain the speed $v_1 = \sqrt{\frac{2GM_E}{R_E}}$ to escape to the Earth

However, before launch it is already moving at 410 m/s to the East because of the Earth rotation. Launching to the East takes advantage of this "free" contribution towards escape speed.

The initial speed v_1 needed for a body to escape from the surface of a spherical body of mass M and radius R (ignoring air resistance is):

$$v_1 = \sqrt{\frac{2GM}{R}} \quad \text{(escape speed)}$$

This equation gives:

$$\begin{cases} v_1 = 5{,}02 \cdot 10^3 \ m/s & \text{MARS} \\ v_1 = 5{,}95 \cdot 10^4 \ m/s & \text{JUPITER} \\ v_1 = 6{,}18 \cdot 10^5 \ m/s & \text{SUN} \end{cases}$$

Black Holes, the Schwarzschild radius, and the Event Horizon

The concept of a black hole is one of the most interesting and startling products of modern gravitational theory, yet the basic idea can be understood on the basis of Newtonian principles.

In the previous problem we have calculated the escape speed from a star of mass M and radius R.

a) Rewrite the escape speed in terms of an average density ρ and planet/star radius R. Calculate the escape velocity from the surface of the Sun ($M=1.99 * 10^{30}$kg, $R=6.96*10^8$m). What fraction represents this with respect to the speed of the light in vacuum ($c=3*10^9$m/s)?

b) The first expression for escape speed $v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}}R$ suggests that any body of mass M will act as a black hole if its radius R is less than or equal to a certain critical radius, called the Schwarzschild radius. From the condition v=c (speed of light) determine this critical radius.

c) Which would be the Schwarzschild radius for the Sun and of the Earth within all their mass shoud be compressed in order to behave as a black hole?

Answer:

a)

ob : $M = \rho V = \rho \frac{4\pi R^3}{3}$ ρ = average density (considered ct)

$$\Rightarrow \boxed{v = \sqrt{\frac{8\pi G\rho}{3}} R} \quad \rightarrow \underline{v \sim R}$$

Consider, now, various stars with same average density ρ an different radii R.

$v \sim R$

For the Sun:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{1.99 \times 10^{30}\,\text{kg}}{\frac{4}{3}\pi(6.96 \times 10^8\,\text{m})^3} = 1410\,\text{kg/m}^3$$

=>

$v = 6.18 \times 10^5\,\text{m/s}$

This represents 1/500 from the speed of light, is independent of the mass of the escaping body; it depends on only the mass and radius (or average density and radius) of the sun.

Consider various stars with the same average density and different radii R. the equation giving the escape speed shows that for a given value of density the escape speed is directly proportional to R. In 1783 the Rev. John Mitchell, an amateur astronomer, noted that if a body with the same average density as the sun had about 500 times the radius of the sun, its escape speed would be greater than the speed of light c. With his statement that "all light emitted from such a body would be made to return toward it," Mitchell became the first person to suggest the existence of what we now call a black hole—an object that exerts a gravitational force on other bodies but cannot emit any light of its own.

b)

From $v = \sqrt{\frac{2GM}{R}}$ we see that a body of mass M acts as a black body if $\underline{R \leq R_{critical}}$

classically R_{crit} can be deduced putting $v = c$

(gives the correct result due to two compensating errors

relativistically the kinetic energy is not $\frac{mv^2}{2}$ and the potential energy of a black hole is not $-\frac{GM}{r}$

So, from:

$$c = \sqrt{\frac{2GM}{R_S}}$$

Solving for the Schwarzschild radius R_S, we find

$$R_S = \frac{2GM}{c^2} \qquad \text{(Schwarzschild radius)}$$

Discussion *If a spherical, nonrotating body with mass M has a radius less than then nothing (not even light) can escape from the surface of the body, and the body is a black hole (Fig.). In this case, any other body within the center of the black hole is trapped by the gravitational attraction of the black hole and cannot escape from it.*

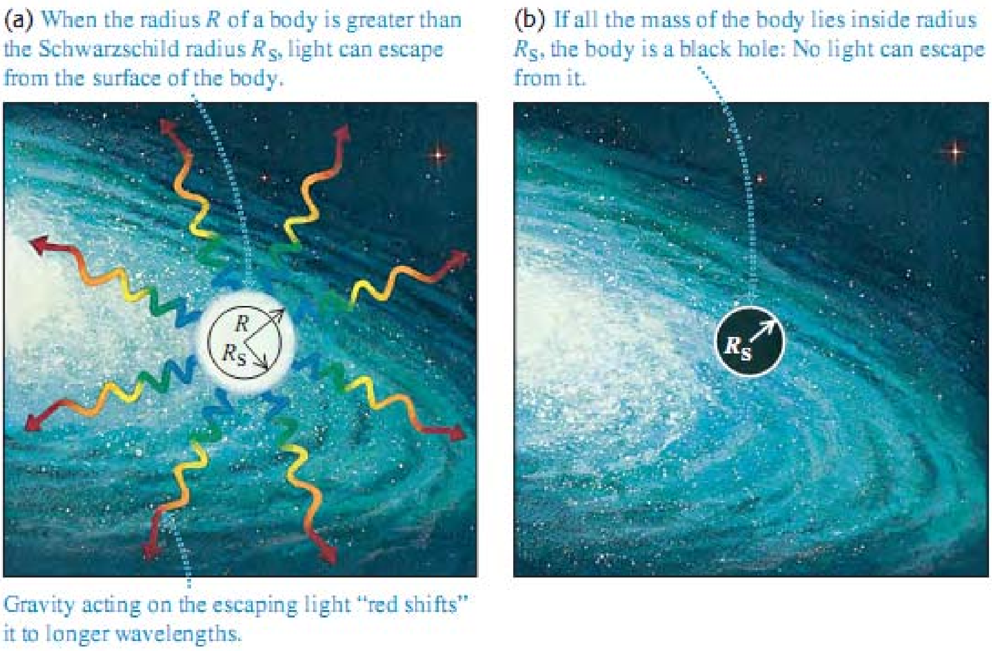
Using generalized theory of relativity one can deduce that

$$\boxed{R_S = \frac{2GM}{c^2}}$$ Schwarzschild radius

If a spherical, nonrotating body with mass M has a radius less than R_S, it will behave as a black hole:

The surface of the sphere with radius R_S surrounding the black hole is called the event horizon: since the light cannot escape within that sphere we can't see the events happening inside!

The sun / earth ~~would~~ behave as a black hole if their radius within all the mass would be compressed would be: $R_S^{sun} = 3\,km$ $R_S^{Earth} = 8{,}7\,mm$



The surface of the sphere with radius surrounding a black hole is called the ***event horizon:*** *Since light can't escape from within that sphere, we can't see events occurring inside. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other bodies), its electric charge (from the electric forces it exerts on other charged bodies), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information about the body is irretrievably lost when it collapses inside its event horizon.*

Observation of black holes

At points far from a black hole, its gravitational effects are the same as those of any normal body with the same mass. If the sun collapsed to form a black hole, the orbits of the planets would be unaffected. But things get dramatically different close to the black hole. If you decided to become a martyr for science and jump into a black hole, the friends you left behind would notice several odd effects as you moved toward the event horizon, most of them associated with effects of general relativity.

If you carried a radio transmitter to send back your comments on what was happening, your friends would have to retune their receiver continuously to lower and lower frequencies, an effect called the gravitational red shift. Consistent with this shift, they would observe that your clocks (electronic or biological) would appear to run more and more slowly, an effect called time dilation. In fact, during their lifetimes they would never see you make it to the event horizon.

In your frame of reference, you would make it to the event horizon in a rather short time but in a rather disquieting way. As you fell feet first into the black hole, the gravitational pull on your feet would be greater than that on your head, which would be slightly farther away from the black hole. The differences in gravitational force on different parts of your body would be great enough to stretch you along the direction toward the black hole and *compress you perpendicular to it. These effects (called tidal forces) would rip you to atoms, and then rip your atoms apart, before you reached the event horizon.*

Observation

All that an observer outside the event horizon can know about a black hole is:

- *its mass (from the gravitational effect on other bodies).*
- *its electric charge (from electric forces that it exerts on other charged bodies).*
- *Its angular momentum (because a rotating black hole tends to drag space and everything in that space around it).*

All the other (direct) information about the black-hole (body with radius smaller than the Schwarzschild radius) is irretrievably lost when it collapses inside the event horizon.

Bibliography

H. D. Young, R. A. Freedman - *Sears and Zemansky's University Physics with Modern Physics Technology Update* (lb. engleza), Pearson - 2013; Chapter 13 Gravitation, p. 402-436.